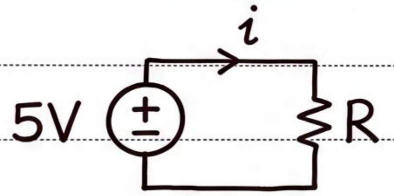
Why do we use imaginary numbers in AC circuit analysis with capacitors and Inductors?

Author: Ahmed Khedr Elsheshtawy

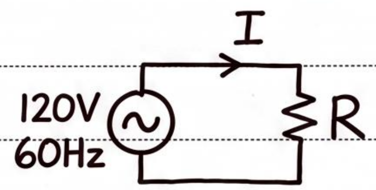
Signature: Ahmed Elsheshtawy.

Why do we use imaginary numbers in AC Circuit analysis?

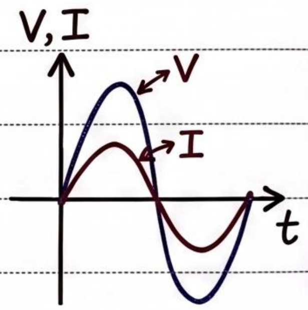
In DC, when having only a resistor: ratio between V and I stays the same at any time t ($R = V/I$), hence V and I are in phase. Same thing occurs in AC when we have only a resistor.



$$i = \frac{V}{R}$$
$$\varnothing = 0$$



$$I = \frac{V}{R}$$
$$\varnothing = 0$$



...With capacitors or inductors, this isn't the case...

$$i = C\frac{dv}{dt} \qquad v = L\frac{di}{dt}$$

Clearly not in phase when using sinusoidal functions (sin, cos)

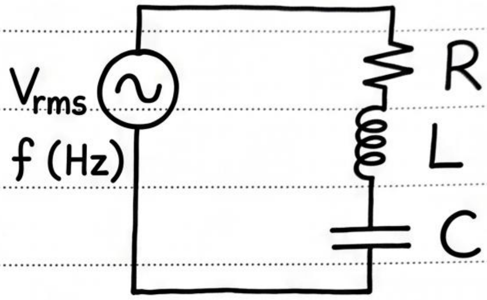
$v = \sin t$

$C = 1 \Rightarrow i = \cos t = \sin(t + 90°)$

Now, we can apply KCL here, but that would be messy.

Applying KVL:

$$V_S = V_R + V_L + V_C$$



$$V_S = V_{rms} \cos(t)$$

$$V_R = V_{Rmax} \cos(t)$$

$$V_L = V_{Lmax} \cos(t + 90°)$$

$$V_C = V_{Cmax} \cos(t - 90°)$$

$$V_{rms} \cos(t) = V_{Rmax} \cos(t) + V_{Cmax} \cos(t - 90°) + V_{Lmax} \cos(t + 90°)$$

→ For a simple circuit, this equation is too difficult to deal with. You can use sine and cosine rules

$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \sin\beta \cos\alpha$$

$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$$

Now to find any unknown (like V_{Rmax}), you would require using trig identities and experience math trauma.

OR, you can use Euler's formula:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

→ any trig expression can be rewritten as exponential.

Also! see: $1 + i = \sqrt{2} \cos(45°) + i\left(\sqrt{2} \sin(45°)\right) = \sqrt{2}\, e^{i(45°)}$

there is always a way to write complex number in this format.

~~For a capacitor. If we say $V = I/2\pi C$, but if we ignore phase for a moment and assume the equation :~~

Where the hell did this come from?

let's say Voltage across a component is $V_{comp} = A\cos(t + \phi_{comp})$, and we want to find the current across it. We can't just multiply V_{comp} by a constant to get the equation of I_{comp} sure multiplying by (-1) can phase shift by 180°, but we don't even know if I is shifted by 180°. So what do we do?

let's ~~see~~ observe this beauty for a second to see if it can benefit us :

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$2e^{i30^\circ} = 2\cos(30^\circ) + 2i\sin(30^\circ)$$

multiply both :

$$e^{i\theta} \times 2e^{i30^\circ} = 2e^{i\theta + i30^\circ} = 2e^{i(\theta+30^\circ)}$$

$$2e^{i(\theta+30^\circ)} = 2\cos(0+30^\circ) + 2\sin(0+30^\circ)$$

by multiplying two functions, we got a new function that has a magnitude of and phase shifted 30°. Surely we can use this in our main problem of $\boxed{V_{comp}}$ and $\boxed{I_{comp}}$ to find $\boxed{I_{comp}}$ across a capacitor or an inductor.

you can also use euler's formula to find the math identity of any sinusoid function.

For example:

$$\cos(2\theta) = ?$$

we can just place that as the angle in euler's formula

$$\Rightarrow e^{2\theta i} = \underbrace{\cos(2\theta)} + i\sin(2\theta)$$

Hey! this is the same function we wanted to find its value!

How can we find it? we can evaluate $e^{2\theta i}$ and find its real component, and that real component would equal $\cos(2\theta)$

$$e^{2\theta} = e^{2(i\theta)} = (e^{i\theta})^2 = (\cos\theta + i\sin\theta)^2$$

$$= \cos^2\theta + 2i\cos\sin\theta - \sin^2\theta \leftarrow i^2 = -1$$

$$= \underbrace{\cos^2\theta - \sin^2\theta}_{\text{real component}} + \underbrace{i\,2\cos\theta\sin\theta}_{\text{imaginary component}}$$

$$\boxed{\text{So, } \cos 2\theta = \cos^2\theta - \sin^2\theta}$$

Also, if we set imaginary components equal to one another: $\boxed{i\sin 2\theta = i\,2\sin\theta\cos\theta}$

Now let's apply that to the capacitor equation : $i = C\frac{du}{dt}$

note : to avoid confusion with current (i), imaginary symbol will be j. $j = \sqrt{-1}$

if $U = A\cos(\omega t + \phi)$, and $C = 1$ (for simplicity)

So, $i = -\omega CA\sin(\omega t + \phi)$

let's instead say :

$U \overset{(Re)}{=} Ae^{j(\omega t+\phi)}$ → makes no sense, but the real component is our voltage

$= A\cos(\omega t + \phi) + jA\sin(\omega t + \phi)$

So, by the end we will see what the real component is

$i = \frac{du}{dt} = \frac{d}{dt}\left(Ae^{j(\omega t+\phi)}\right) = j\omega C\underbrace{Ae^{j(\omega t+\phi)}}$

So, we can say, $\boxed{i = j\omega C \cdot \overline{V}}$ *

$i = j\omega C \times (A\cos(\omega t+\phi) + jA\sin(\omega t+\phi))$

$= \boxed{-\omega CA\sin(\omega t+\phi)} + j\omega CA\cos(\omega t+\phi))$

real component is still i as we got normally by deriving $\left(\frac{du}{dt}\right)$ but ! It + gave us an equation of that is equivalent but doesn't involve differentiation.

$$\boxed{i = j\omega C\,\overline{V}} \Leftrightarrow \boxed{i = C\frac{du}{dt}}$$

Now we can rearrange $i = j\omega C V$ to:

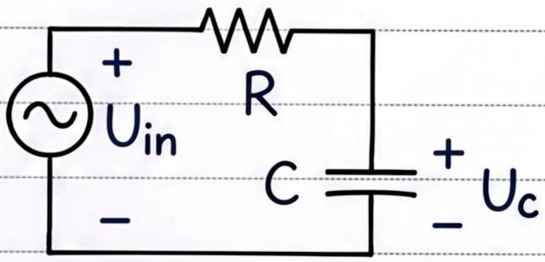
$$\frac{v}{i} = \frac{1}{j\omega C} = X_c$$

which is just the impedance of the capacitor to current.

Notice: $R = \frac{U}{i}$, and $X_c = \frac{U}{I}$

and the j of the imaginary numbers accounts for a phase shift of 90° just like cos and sin.

Now let's put a real example:



applying KVL:

$$U_{in} = U_c + U_R$$

$$U_R = iR = C \cdot \frac{dU_c}{dt} \cdot R = RC \cdot \frac{dU_c}{dt}$$

$$U_{in} = U_c + RC \cdot \frac{dU_c}{dt}$$

we know U_c would be of the form $U_c = A\cos(\omega t + \theta)$, but instead of substituting that, which wouldn't be fun, we can use the earlier function we came up with.

$$U_c = Ae^{j(\omega t + \theta)}, \text{ and } \frac{dU_c}{dt} = j\omega U_c$$

So, $U_{in} = U_c + RC(j\omega U_c) = U_c(1 + j\omega RC)$

and we finally get: $\boxed{U_c = \frac{1}{1 + j\omega RC} \cdot U_{in}}$

→ cont.

For simplicity, $\omega = C = R = 1$

$$V_c = \frac{1}{1+j} V_i = \frac{1}{1+j} \times \frac{1-j}{1-j} V_i$$

$$= \frac{1-j}{1-j^2} V_i = \frac{1-j}{1+1} V_i = \frac{1}{2} V_i - \frac{1}{2} V_i$$

$$\boxed{V_c = \frac{1}{2} V_i - \frac{1}{2} j V_i}$$

so, if $V_i = 3\cos(t + 10°)$

we can use euler's formula, $V_i = 3e^{j(t+10°)}$

note: while putting in mind that the actual voltage of $V_i = 3e^{j(t+10°)}$ is its real component.

$$V_c = \frac{\sqrt{2}}{2} e^{j(-45°)} V_i$$

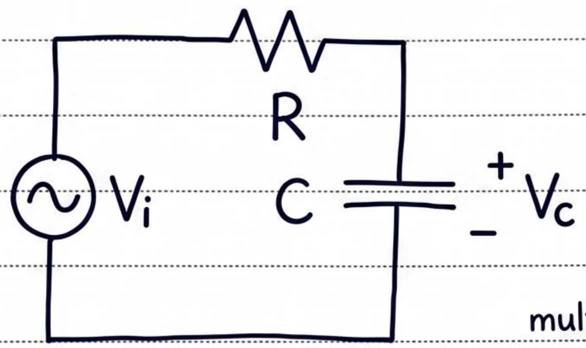
$$V_c = \frac{\sqrt{2}}{2} e^{j(-45°)} * (3e^{j(t+10)})$$

$V_c = \dfrac{3\sqrt{2}}{2} e^{j(t-35)}$, but of course we only care about the real component.

$$\boxed{V_c = \frac{3\sqrt{2}}{2} \cos(t - 35°)}$$

this result is how imaginary numbers help us.

And now going back to the original circuit:



using our earlier equation:

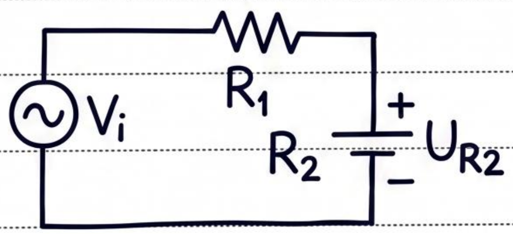
$$U_c = \frac{1}{1 + jwRC} V_i$$

multiply top and bottom by $\frac{1}{jwc}$, which is impedance of C.

$$U_c = \frac{1}{1 + jwRC} \cdot U_i \times \frac{\frac{1}{jwc}}{\frac{1}{jwc}} = \frac{\frac{1}{jwc}}{\frac{1}{jwc} + R} V_i$$

Notice: $$\boxed{U_c = \frac{\frac{1}{jwc}}{\frac{1}{jwc} + R} V_i}$$

that is very similar to the voltage divider equation.



$$U_{R2} = \frac{R_2}{R_1 + R_2} V_i$$

So, we can treat impedance of Capacitor $\left(\frac{1}{jwc}\right)$ like it is a resistance. $X_c = \frac{1}{jwc}$

so we can apply series and parallel resistance Theorems, voltage and current dividers, but the only difference there will be complex numbers and phase shifts.

imaginary

Now:

After studying please solve problems on AC circuit analysis and use with it Euler's Formula to let consepts sink in.